Combining the k-CNF and XOR Phase-Transitions

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Random k-CNF Satisfiability [Franco and Paull, 1983]

- **Definition:** Let $\text{CNF}_k(n, r)$ be a random variable denoting a uniformly chosen $k$-CNF formula with $n$ variables and $\lceil nr \rceil$ $k$-CNF clauses.
  - $n$: The number of variables.
  - $k$: The width of every CNF clause.
  - $r$: CNF clause density $= \text{Ratio of } \# \text{ of CNF clauses to } \# \text{ of variables.}$

- **Ex:** $(X_1 \lor \neg X_5 \lor X_6) \land (\neg X_1 \lor X_3 \lor X_5)$ is one possible value for $\text{CNF}_3(6, 1/3)$.

- **Problem:** Fixing $k$ and $r$, what is the asymptotic probability that $\text{CNF}_k(n, r)$ is satisfiable as $n$ goes to infinity?
k-CNF Phase Transition

Probability that $\text{CNF}_3(400, r)$ is satisfiable

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**k-CNF Phase Transition**

Probability that $\text{CNF}_3(400, r)$ is satisfiable

![Graph showing the probability of satisfiability for CNF_3(400, r)]

**k-CNF Phase-Transition Conjecture:**
For every $k \geq 2$, there is a constant $r_k > 0$ such that:

$$\lim_{n \to \infty} \Pr(\text{CNF}_k(n, r) \text{ is sat.}) = \begin{cases} 
1 & \text{if } r < r_k \\
0 & \text{if } r > r_k 
\end{cases}$$
**XOR Phase-Transition** [Creignou and Daudé, 1999]

- **Definition:** An XOR clause is the exclusive-or of a set of variables, possibly including 1 as well.
  
  Ex: $X_2 \oplus X_4, 1 \oplus X_1 \oplus X_2 \oplus X_7$

- **Definition:** Let $\text{XOR}(n, s)$ be a random variable denoting a uniformly chosen XOR formula with $n$ variables and $\lceil ns \rceil$ XOR clauses.
  
  - $n$: The number of variables.
  - $s$: XOR clause density = Ratio of # of XOR clauses to # of variables.

**Problem:** Fixing $s$, what is the asymptotic probability that $\text{XOR}(n, s)$ is satisfiable as $n$ goes to infinity?
XOR Phase-Transition [Creignou and Daudé, 1999]

• **Definition:** An XOR clause is the exclusive-or of a set of variables, possibly including 1 as well.
  
  **Ex:** $X_2 \oplus X_4, 1 \oplus X_1 \oplus X_2 \oplus X_7$

• **Definition:** Let $\text{XOR}(n, s)$ be a random variable denoting a uniformly chosen XOR formula with $n$ variables and $[ns]$ XOR clauses.
  
  • $n$: The number of variables.
  • $s$: XOR clause density = Ratio of # of XOR clauses to # of variables.

**Problem:** Fixing $s$, what is the asymptotic probability that $\text{XOR}(n, s)$ is satisfiable as $n$ goes to infinity?

$$\lim_{n \to \infty} \Pr(\text{XOR}(n, s) \text{ is sat.}) = \begin{cases} 
1 & \text{if } s < 1 \\
0 & \text{if } s > 1 
\end{cases}$$
Combining k-CNF and XOR Together

• **Motivation:** Hashing-based sampling and counting algorithms use formulas with both k-CNF and XOR clauses.
  - [Gomes et al. 2007], [Chakraborty et al., 2013], [Ermon et al. 2013]

• **Definition:** A **k-CNF-XOR formula** is the conjunction of k-CNF and XOR clauses.

• **Goal:** Analyze the “behavior” of k-CNF-XOR formulas.
  - In this work we analyze the asymptotic satisfiability of random k-CNF-XOR formulas.
Random k-CNF-XOR Satisfiability

- **Definition**: Let $\psi_k(n, r, s)$ be a random variable denoting $\text{CNF}_k(n, r) \land \text{XOR}(n, s)$
  - i.e. the conjunction of $[nr]$ random $k$-CNF clauses and $[ns]$ random XOR clauses.
  - $n$: The number of variables.
  - $k$: The width of every CNF clause.
  - $r$: $k$-CNF clause density.
  - $s$: XOR clause density.

**Problem**: Fixing $k$, $r$, and $s$, what is the asymptotic probability that $\psi_k(n, r, s)$ is satisfiable as $n$ goes to infinity?
k-CNF-XOR: What Do We Expect to See?

Probability that $\psi_5(n, r, s) = \text{CNF}_5(n, r) \land \text{XOR}(n, s)$ is satisfiable
Probability that $\psi_5(100, r, s) = \text{CNF}_5(100, r) \land \text{XOR}(100, s)$ is satisfiable
Theorem 1: The k-CNF-XOR Phase-Transition Exists

\[ \psi_k(n, r, s) = \text{CNF}_k(n, r) \land \text{XOR}(n, s) \] is a random variable denoting a uniformly chosen \( k \)-CNF-XOR formula over \( n \) variables with CNF-density \( r \) and XOR-density \( s \).

**Thm 1:** For all \( k \geq 2 \), there are functions \( \phi_k \) and constants \( \alpha_k \geq 1 \) such that random \( k \)-CNF-XOR formulas have a phase-transition located at \( s = \phi_k(r) \) when \( r < \alpha_k \).

For all \( s \geq 0 \), and \( 0 \leq r \leq \alpha_k \) (except for at most countably many \( r \)):

\[
\lim_{n \to \infty} \Pr(\psi_k(n, r, s) \text{ is sat.}) = \begin{cases} 
1 & \text{if } s < \phi_k(r) \\
0 & \text{if } s > \phi_k(r) 
\end{cases}
\]

What can we say about \( \phi_k \)?
Theorem 2: Locating the Phase-Transition

What can we say about $\phi_k$, the location of the $k$-CNF-XOR phase-transition?

**Thm 2:** For $k \geq 3$, we have linear upper and lower bounds on $\phi_k(r)$.
Conclusion

• There is a phase-transition in the satisfiability of random k-CNF-XOR formulas at k-CNF clause densities below $\alpha_k$.

• We have some explicit bounds on the location.

Future Work:

• **Conjecture**: There is a phase-transition in k-CNF-XOR formulas at all k-CNF clause densities.

• **Conjecture**: $\phi_k(r)$ is linear for k-CNF clause densities below some $\alpha_k^* > 0$.

• How does the runtime of SAT solvers on k-CNF-XOR equations behave near the phase-transition?
Thanks!


Runtime Behavior at the Transition

Average satisfiability and solve time of $F_3(200, 200r)$

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