Combining the k-CNF and XOR Phase-Transitions

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Random k-CNF Satisfiability [Franco and Paull, 1983]

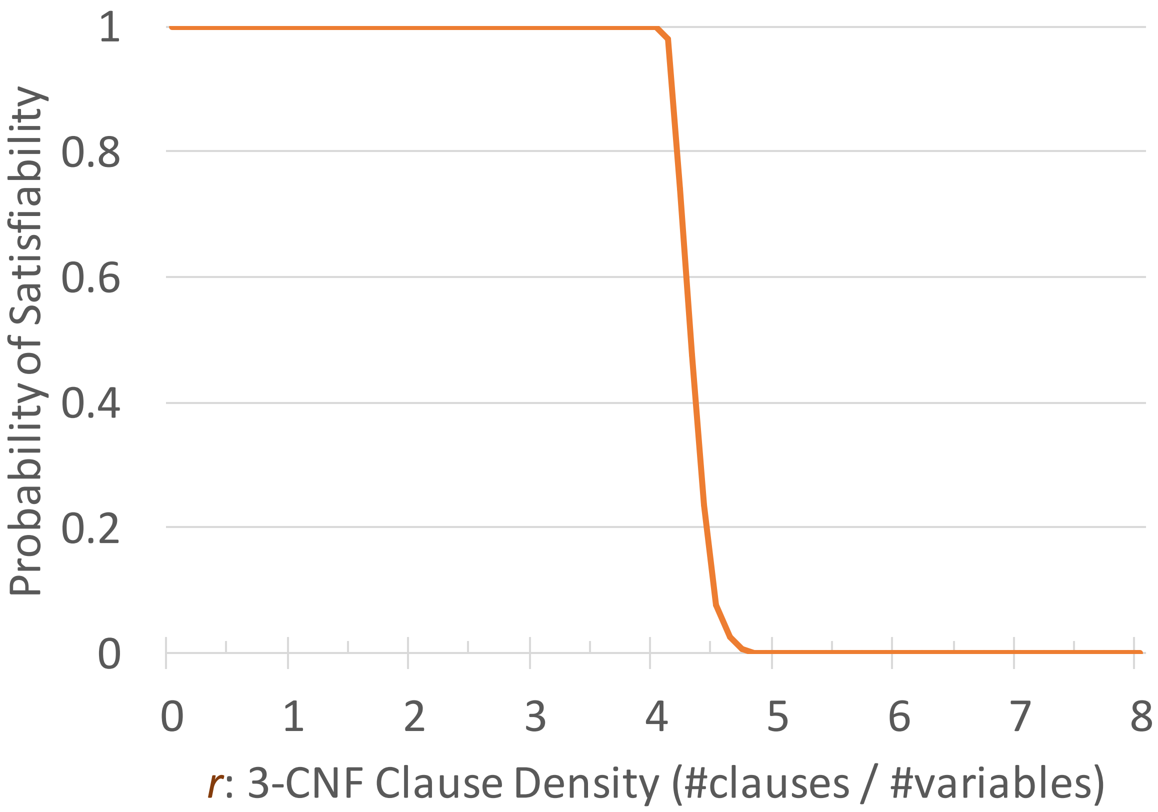
- **Definition:** Let $\mathbf{CNF}_k(\boldsymbol{n}, \boldsymbol{r})$ be a random variable denoting a uniformly chosen k-CNF formula with n variables and $\lceil nr \rceil$ k-CNF clauses.
  - $\boldsymbol{n}$: The number of variables.
  - $\boldsymbol{k}$: The width of every CNF clause.
  - $\boldsymbol{r}$: CNF clause density = Ratio of # of CNF clauses to # of variables.

- **Ex:** $(X_1 \vee \neg X_5 \vee X_6) \wedge (\neg X_1 \vee X_3 \vee X_5)$ is one possible value for $\text{CNF}_3(6, 1/3)$.

- **Problem:** Fixing k and r, what is the asymptotic probability that $\text{CNF}_k(n, r)$ is satisfiable as n goes to infinity?

k-CNF Phase Transition

Probability that $\mathrm{CNF}_3(400, r)$ is satisfiable



k-CNF Phase Transition

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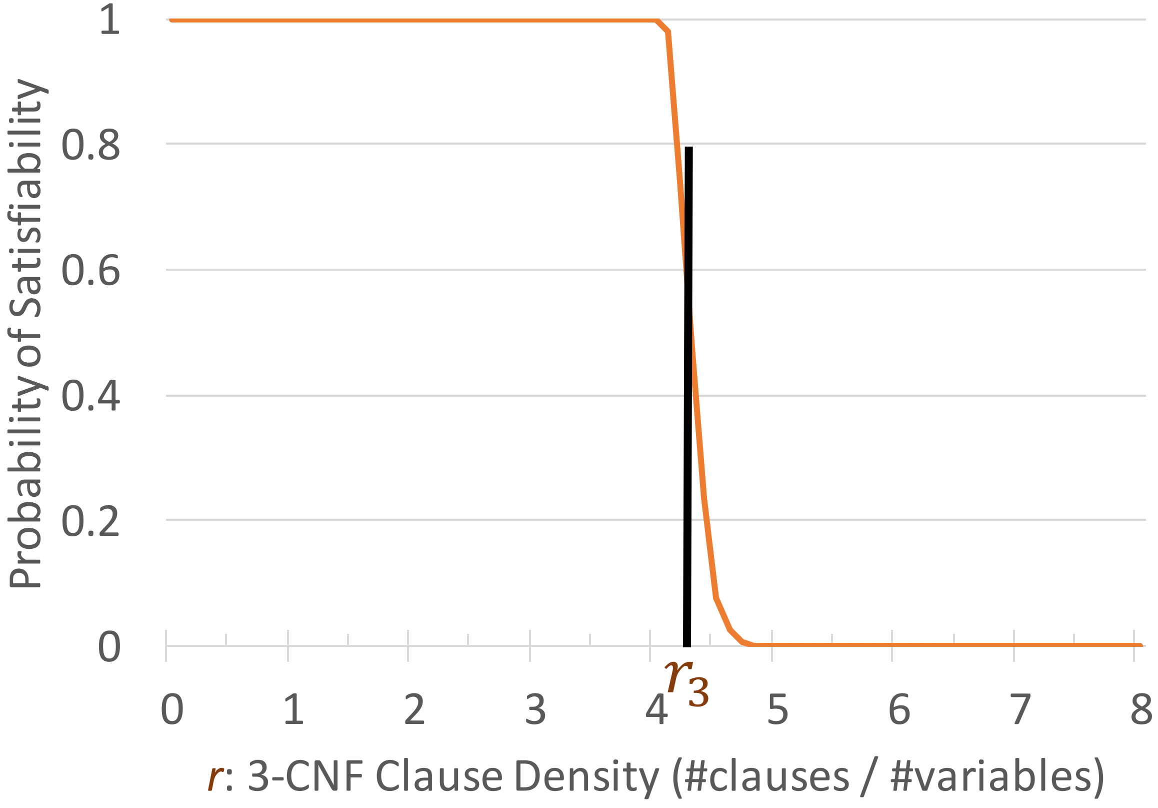

k-CNF Phase-Transition Conjecture:
For every $k \geq 2$, there is a constant $r_k > 0$ such that:

$$\lim_{n \to \infty} \Pr(\text{CNF}_k(n, r) \text{ is sat.}) = \begin{cases} 1 & \text{if } r < r_k \\ 0 & \text{if } r > r_k \end{cases}$$

XOR Phase-Transition [Creignou and Daudé, 1999]

- **Definition:** An **XOR clause** is the *exclusive-or* of a set of variables, possibly including 1 as well.

  Ex: $X_2 \oplus X_4$, $1 \oplus X_1 \oplus X_2 \oplus X_7$
- **Definition**: Let $\mathbf{XOR}(\boldsymbol{n}, \boldsymbol{s})$ be a random variable denoting a uniformly chosen XOR formula with n variables and $\lceil ns \rceil$ XOR clauses.
  - $\boldsymbol{n}$: The number of variables.
  - $\boldsymbol{s}$: XOR clause density = Ratio of # of XOR clauses to # of variables.

Problem: Fixing s, what is the asymptotic probability that $\mathrm{XOR}(n, s)$ is satisfiable as n goes to infinity?

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Problem: Fixing s, what is the asymptotic probability that $\mathrm{XOR}(n, s)$ is satisfiable as n goes to infinity?

$$\lim_{n \to \infty} \Pr(\mathrm{XOR}(n, s) \text{ is sat.}) = \begin{cases} 1 & \text{if } s < 1 \\ 0 & \text{if } s > 1 \end{cases}$$

Combining k-CNF and XOR Together

- **Motivation:** Hashing-based sampling and counting algorithms use formulas with ***both*** k-CNF and XOR clauses.
  - [Gomes *et al.* 2007], [Chakraborty *et al.*, 2013], [Ermon *et al.* 2013]

- **Definition:** A **k-CNF-XOR formula** is the conjunction of k-CNF and XOR clauses.

- **Goal**: Analyze the "behavior" of k-CNF-XOR formulas.
- In this work we analyze the asymptotic satisfiability of random k-CNF-XOR formulas.

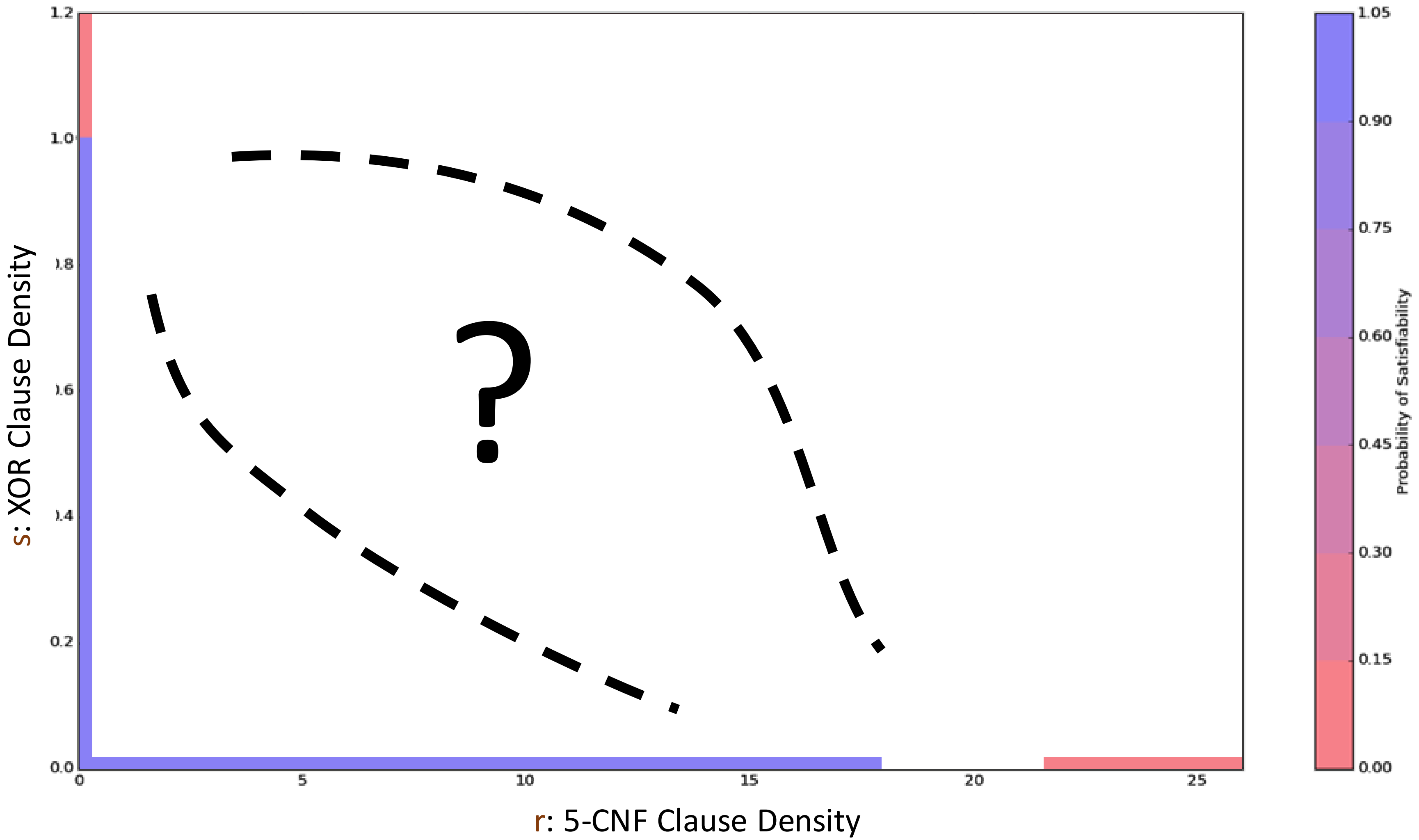
Random k-CNF-XOR Satisfiability

- **Definition:** Let $\psi_k(n, r, s)$ be a random variable denoting $\mathrm{CNF}_k(n, r) \wedge \mathrm{XOR}(n, s)$
  - i.e. the conjunction of $\lceil nr \rceil$ random k-CNF clauses and $\lceil ns \rceil$ random XOR clauses.
  - n: The number of variables.
  - k: The width of every CNF clause.
  - r : k-CNF clause density.
  - s : XOR clause density.

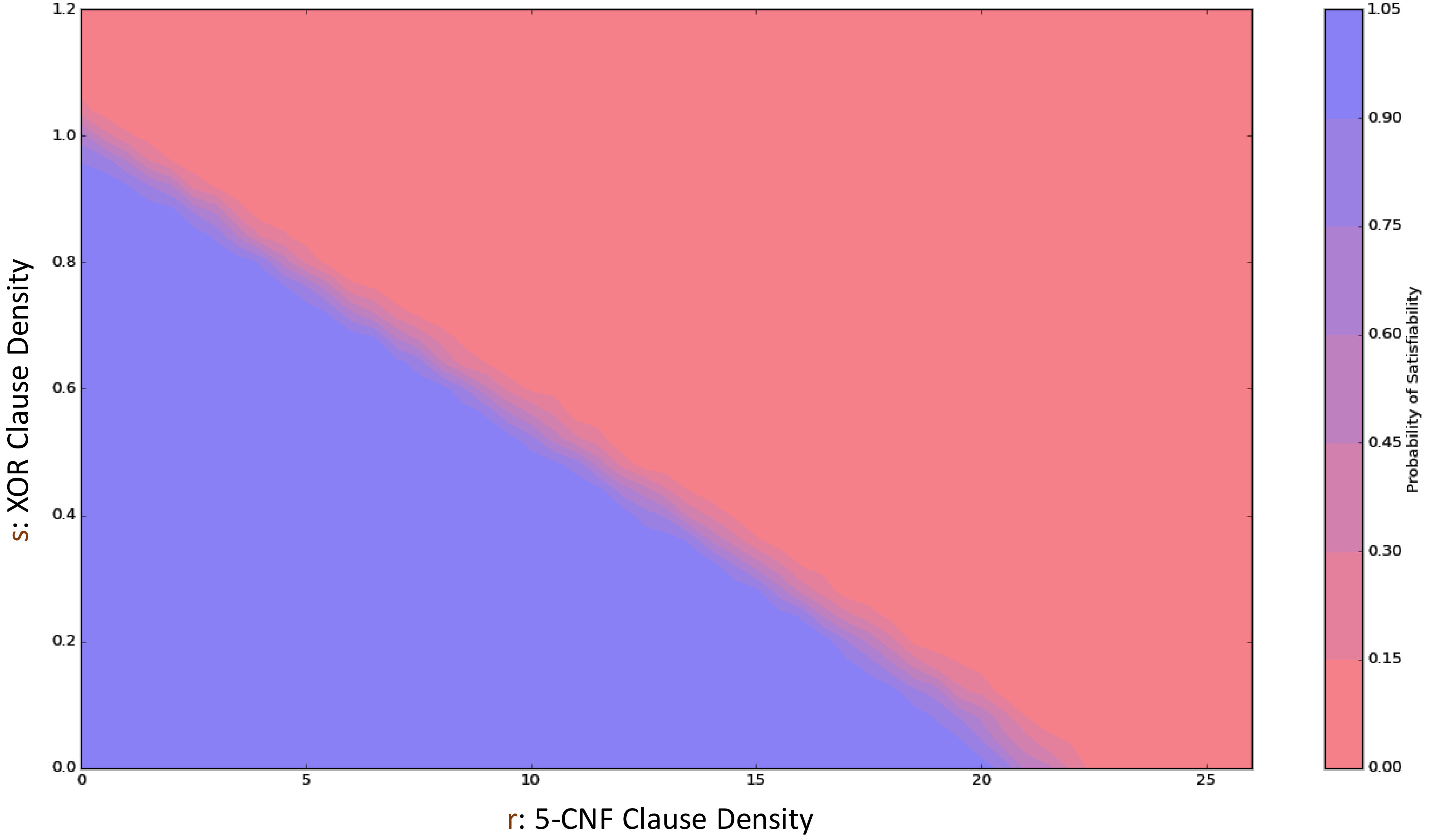
Problem: Fixing k, r, and s, what is the asymptotic probability that $\psi_k(n, r, s)$ is satisfiable as n goes to infinity?

k-CNF-XOR: What Do We Expect to See?

Probability that $\psi_5(n, r, s) = \mathrm{CNF}_5(n, r) \wedge \mathrm{XOR}(n, s)$ is satisfiable



Probability that $\psi_5(100, r, s) = \mathrm{CNF}_5(100, r) \wedge \mathrm{XOR}(100, s)$ is satisfiable



Theorem 1: The k-CNF-XOR Phase-Transition Exists

$\psi_k(n, r, s) = \text{CNF}_k(n, r) \wedge \text{XOR}(n, s)$ is a random variable denoting a uniformly chosen k-CNF-XOR formula over n variables with CNF-density r and XOR-density s.

Thm 1: For all $k \geq 2$, there are functions ϕ_k and constants $\alpha_k \geq 1$ such that random k-CNF-XOR formulas have a phase-transition located at $s = \phi_k(r)$ when $\mathrm{r} < \alpha_k$.

For all $s \geq 0$, and $0 \leq r \leq \alpha_k$ (except for at most countably many r):

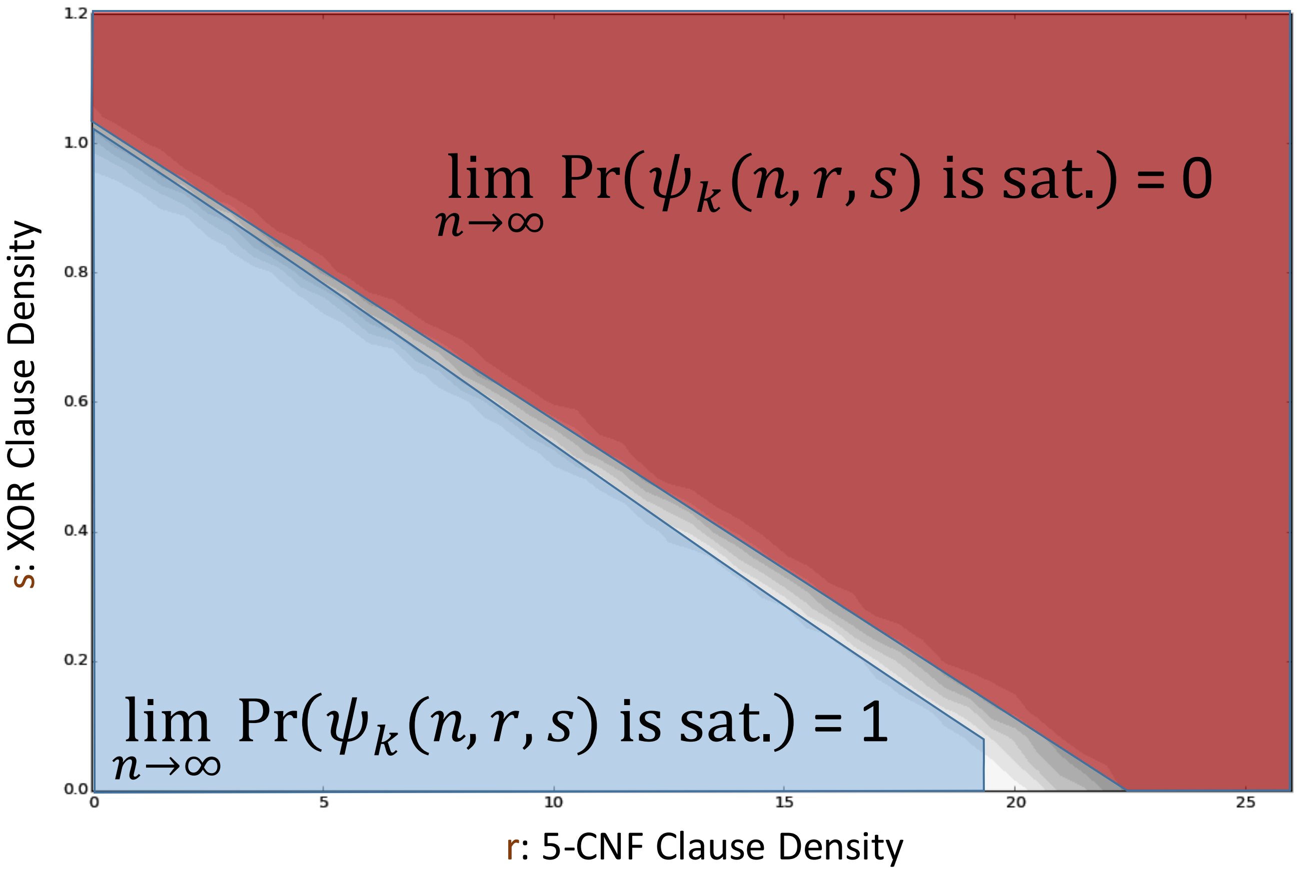
$$\lim_{n \to \infty} \Pr(\psi_k(n, r, s) \text{ is sat.}) = \begin{cases} 1 & \text{if } s < \phi_k(r) \\ 0 & \text{if } s > \phi_k(r) \end{cases}$$

What can we say about ϕ_k?

Theorem 2: Locating the Phase-Transition

What can we say about ϕ_k, the location of the k-CNF-XOR phase-transition?

Thm 2: For $k \geq 3$, we have linear upper and lower bounds on $\phi_k(r)$.



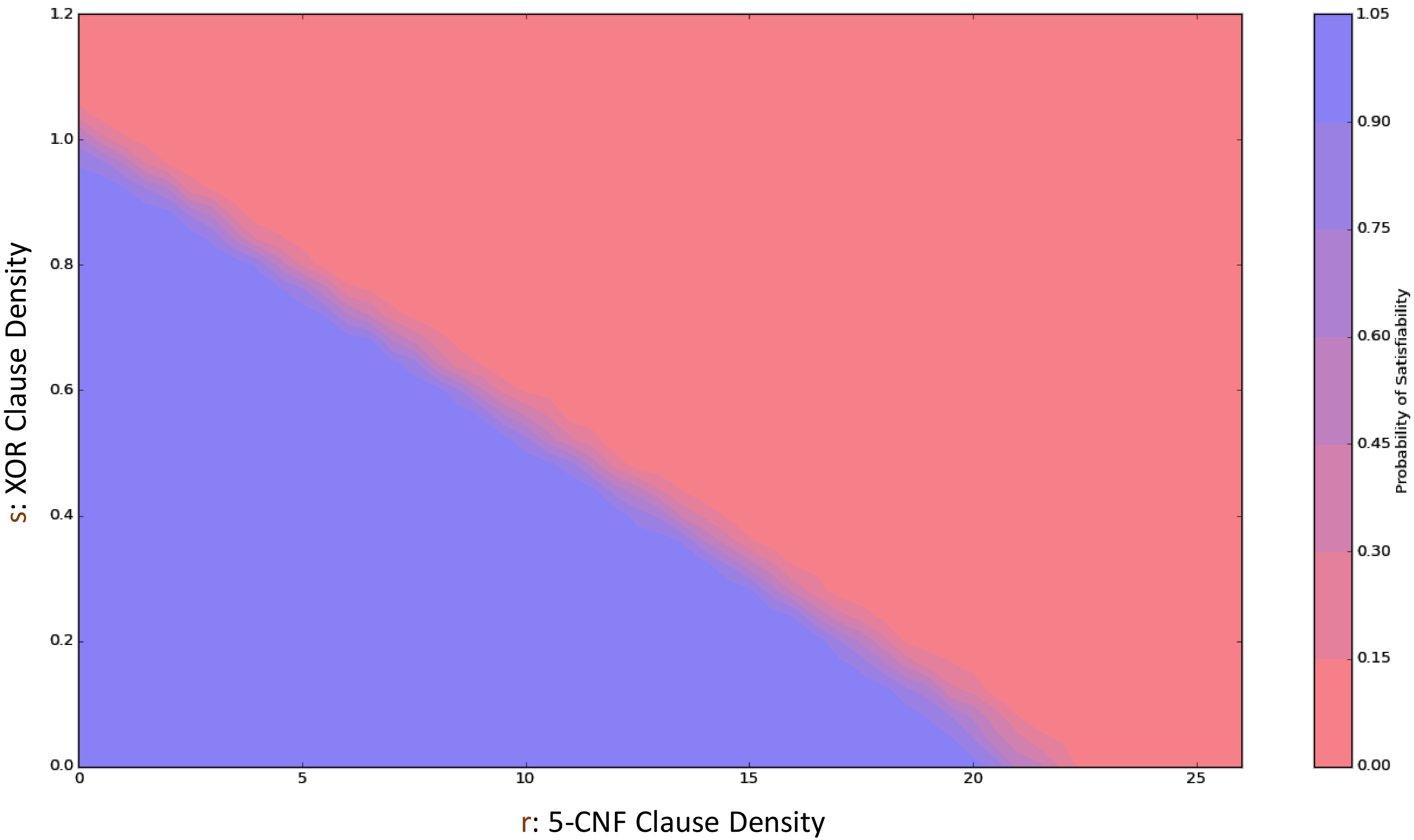
Conclusion

- There is a phase-transition in the satisfiability of random k-CNF-XOR formulas at k-CNF clause densities below α_k.
- We have some explicit bounds on the location.

Future Work:

- **Conjecture**: There is a phase-transition in k-CNF-XOR formulas at all k-CNF clause densities.
- **Conjecture**: $\phi_k(r)$ is linear for k-CNF clause densities below some $\alpha_k^* > 0$.
- How does the runtime of SAT solvers on k-CNF-XOR equations behave near the phase-transition?

Thanks!



1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
5
10
15
20
25
1.05
0.90
0.75
0.60
0.45
0.30
0.15
0.00
Probability of Satisfiability
s: XOR Clause Density
r: 5-CNF Clause Density

Citations

- [Ermon *et al.* 2013] S. Ermon, C. P. Gomes, A. Sabharwal, and B. Selman. Taming the curse of dimensionality: Discrete integration by hashing and optimization. In *Proc. of ICML*, pages 334–342, 2013.
- [Franco and Paull, 1983] J. Franco and M. Paull. Probabilistic analysis of the Davis– Putnam procedure for solving the satisfiability problem. *Discrete Applied Math*ematics, 5(1):77–87, 1983.
- [Chakraborty *et al.* 2013] S. Chakraborty, K. S. Meel, and M. Y. Vardi. A scalable and nearly uniform generator of SAT witnesses. In *Proc. of CAV*, pages 608–623, 2013.
- [Creignou and Daudé, 1999] N. Creignou and H. Daudé. Satisfiability threshold for random xor-cnf formulas. *Discrete Applied Mathematics*, 9697:41 – 53, 1999.
- [Gomes *et al.* 2007] C.P. Gomes, A. Sabharwal, and B. Selman. Near-Uniform sampling of combinatorial spaces using XOR constraints. In *Proc. of NIPS*, pages 670–676, 2007
- [Goerdt, 1996] A. Goerdt. A threshold for unsatisfiability. *Journal of Computer and System* Sciences, 53(3):469 – 486, 1996.

Runtime Behavior at the Transition